

ON SMOOTH MANIFOLDS WITH HOMOTOPY TYPE OF A HOMOLOGY SPHERE

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ABSTRACT. Let X be a \mathbb{Z}/n -homology m -sphere with a map $f : S^m \rightarrow X$ inducing the isomorphism of \mathbb{Z}/n -homologies. Assume that m and the order of $\pi_1(X)$ are odd. In this paper, we prove a lower bound theorem for the number of diffeomorphism classes of manifolds that are simple homotopy equivalent to X , under certain conditions on n and m .

1. INTRODUCTION

Unless otherwise stated, by a manifold we mean a smooth, oriented, closed manifold with dimension greater than or equal to 5. Given a simple Poincaré complex X with formal dimension m , a classical problem in topology is to understand the set of diffeomorphism classes of smooth manifolds in the simple homotopy type of X . For such an aim, a fundamental object to study is the smooth simple structure set $\mathcal{S}^s(X)$, see [10] page 125-126 for notation and details. Elements of $\mathcal{S}^s(X)$ are equivalence classes of simple homotopy equivalences $\omega : M \rightarrow X$ from an m -dimensional manifold M . Two such homotopy equivalences $\omega_1 : M_1 \rightarrow X$ and $\omega_2 : M_2 \rightarrow X$ are said to be equivalent if there is a diffeomorphism $g : M_1 \rightarrow M_2$ so that the triangle commutes up to homotopy. Note that composition of an element in $\mathcal{S}^s(X)$ with a simple self equivalence of X will give another element in $\mathcal{S}^s(X)$ although the manifold is still the same. Hence we need to quotient out such elements in order to get the set of diffeomorphism classes of smooth manifolds in the simple homotopy type of X . Denote $\text{Aut}_s(X)$ the group of homotopy classes of simple self equivalences of X . Then $\text{Aut}_s(X)$ acts on $\mathcal{S}^s(X)$ by composition. The set of diffeomorphism classes of smooth manifolds in the simple homotopy type of X , $\mathcal{M}(X)$, is defined as the set of orbits of $\mathcal{S}^s(X)$ under the action of $\text{Aut}_s(X)$, i.e. $\mathcal{M}(X) := \mathcal{S}^s(X) / \text{Aut}_s(X)$.

Let $K(X)$ denote the group of homotopy classes of maps $[X, BSO]$. It is well known that $K(-)$ is a cohomology theory, see [1]. Let $\text{Aut}_s(K(X))$ denote the subgroup of $\text{Aut}(K(X))$ that consist of automorphisms induced by the simple self equivalences of X . There is a canonical action of $\text{Aut}_s(K(X))$ on $K(X)$ again given by composition. We denote by $\mathfrak{K}(X)$ the set of orbits of $K(X)$ under the action of $\text{Aut}_s(K(X))$. As pointed out in [10] computation of $\text{Aut}_s(X)$ and $\mathcal{S}^s(X)$ is in general difficult, so computation of $\mathcal{M}(X)$ is as well. On the other hand, the computation of $K(X)$ and $\text{Aut}_s(K(X))$ is possible in most cases as $K(-)$ is a cohomology theory. In this paper we compare $\mathfrak{K}(X)$ with $\mathcal{M}(X)$ for certain cases X . In particular, there is a map $\Psi : \mathcal{M}(X) \rightarrow \mathfrak{K}(X)$, see Proposition 3.7, defined by the pullbacks of normal bundles of manifolds in $\mathcal{M}(X)$. Our purpose

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is to determine the image of this map. Notice that one can get information about $\text{Aut}_s(X)$ or $\mathcal{S}^s(X)$ from the image of this map.

The method we use rely on the differentials in the James spectral sequence, see [17] Section II for James spectral sequence. Hence, we need to use some characteristic classes arguments to determine the image of Ψ , see Section 3 and in particular Lemma 3.3. Let $p = 2b + 1$ be an odd prime. For any vector bundle ξ over X , there exist cohomology classes $q_k^p(\xi)$ in $H^{4bk}(X; \mathbb{Z}/p)$, known as mod p Wu classes, introduced in [22]. We write q_k instead of q_k^p if the prime we consider is clear from the context. These classes are defined by the identity $q_k(\xi) = \theta^{-1} \mathcal{P}^k \theta(1)$. Here, \mathcal{P}^n denote the Steenrod's reduced p -th power operation and $\theta : H^*(X; \mathbb{Z}/p) \rightarrow H^*(T\xi; \mathbb{Z}/p)$ denote the Thom isomorphism. For more details on these Wu classes we refer [13], Ch.19.

For each prime p , let $q_1^p(X)$ denote the negative of the mod p Wu class of the Spivak normal bundle of X [16]. Given S a set of primes, let $K(X)_{(S, q_1)}$ denote the subset of $K(X)$ that consist of elements ξ such that for each p in S the first mod p Wu class of ξ satisfy the identity $q_1^p(\xi) + q_1^p(X) = 0$. Since the class $q_1^p(X)$ is a homotopy type invariant of X , see [13], then the subset $K(X)_{(S, q_1)}$ is invariant under the action of $\text{Aut}_s(K(X))$. We denote the quotient of this action by $\mathfrak{K}(X)_{(S, q_1)}$. In particular, if $S = \emptyset$ then $\mathfrak{K}(X)_{(S, q_1)} = \mathfrak{K}(X)$.

For a given prime p , by a \mathbb{Z}/p -homology m -sphere we mean a simple Poincaré complex X with \mathbb{Z}/p -homology of S^m , i.e. $H_*(X; \mathbb{Z}/p) = H_*(S^m; \mathbb{Z}/p)$. We also assume X is p -good, that is the p -completion X_p^\wedge is p -complete, see Definition I.5.1 and Proposition I.5.2 in [4]. The main object of this paper is to prove following:

Theorem 1.1. *Let m be an odd number and S be a subset of the set of primes between $(m+4)/4$ and $(m+2)/2$. For every prime $q < (m+2)/2$ with $q \notin S$, let X be a \mathbb{Z}/q -homology m -sphere with a given \mathbb{Z}/q -homology isomorphism $f : S^m \rightarrow X$, whose fundamental group is of odd order. Then the image of Ψ consists of orbits in $\mathfrak{K}(X)_{(S, q_1)}$ that are represented by elements in the kernel of $f^* : K(X) \rightarrow K(S^m)$. In particular, if $S = \emptyset$ and $m \not\equiv 1 \pmod{8}$ then Ψ is surjective.*

Note that this result help us to put a lower bound on the number of diffeomorphism classes of manifolds which are simple homotopy equivalent to such X . Some examples of complexes that is covered by the assumptions of the theorem are manifolds whose universal cover is a sphere and whose fundamental group has order relatively prime to n , provided the assumptions on the dimension holds. Manifolds in the homotopy types of lens spaces, so called fake lens spaces, are most well known examples. Some applications of our theorem are discussed in Section 3.2.

It is well-known that a degree one normal map $f : M \rightarrow X$ can be surgered to a simple homotopy equivalence if and only if the associated surgery obstruction vanishes (see [21]). The main results in [2] states that if $\pi_1(X)$ is of odd order then the odd dimensional (simple) surgery obstruction groups, $L_m^s(\mathbb{Z}[\pi_1(X)])$, vanish, i.e. every degree one normal map can be surgered to a simple homotopy equivalence. The bottom line of the above theorem is that under the given conditions, elements in the kernel of f^* admitting a degree one map are the ones having same mod p Wu class as the Spivak normal bundle, for the primes $p \in S$. In particular, if $m \not\equiv 1 \pmod{8}$ every element satisfying the relation $q_1^p(\xi) + q_1^p(X) = 0$ for $p \in S$ admits a degree one normal map. If the case when $S = \emptyset$ and $m \not\equiv 1 \pmod{8}$ every stable vector bundle $X \rightarrow BSO$ admits a degree one normal map.

2. PRELIMINARIES

Let X be a simple Poincaré complex with formal dimension m , see [20] for Poincaré complexes. Given a stable vector bundle $\xi : X \rightarrow BSO$, denote $\Omega_k(\xi)$ the cobordism group of k -dimensional manifolds, whose stable normal bundles lift to X through ξ . An element of $\Omega_k(\xi)$ is often denoted by $[\nu : M \rightarrow X]$, where M is a k -manifold and ν is a lifting of its stable normal bundle to X through ξ , and brackets denote the homotopy class of such liftings (see [17] Proposition 2 for notation). Such a lifting is called a *normal map*, and if this map has degree one then it is called a *degree one normal map*, see [11] Definition 3.46. Due to the Pontrjagin-Thom construction $\Omega_k(\xi)$ is isomorphic to k -th homotopy group of $M\xi$, the Thom spectra associated to ξ [18]. Our primary tool is the James spectral sequence, which is a variant of the Atiyah-Hirzebruch spectral sequence.

Let h be a generalized homology theory represented by a connective spectrum, $F \rightarrow X \xrightarrow{f} B$ be an h -orientable fibration with fiber F and $\xi : X \rightarrow BSO$ be a stable vector bundle. The James spectral sequence for h , f and ξ has E_2 -page $E_{s,t}^2 = H_s(B; h_t(M\xi|_F))$ and converges to $h_{s+t}(M\xi)$ (see [17], Section II). In the case when h is the stable homotopy, i.e. represented by the sphere spectrum, the edge homomorphism of this spectral sequence coming from the base line is given as follows:

Proposition 2.1 (see [17] Proposition 2). *The edge homomorphism of the James spectral sequence for stable homotopy, $f : X \rightarrow B$ and $\xi : X \rightarrow BSO$ is a homomorphism $\epsilon\partial : \Omega_n(\xi) \rightarrow H_n(B, \mathbb{Z})$ given by*

$$\epsilon\partial[\nu : M \rightarrow X] = f_* \circ \nu_*[M]$$

for every element $[\nu : M \rightarrow X] \in \Omega_n(\xi)$.

The Atiyah-Hirzebruch spectral sequences for $M\xi$ is isomorphic to the James spectral sequence for stable homotopy, $id : X \rightarrow X$ and $\xi : X \rightarrow BSO$. This follows from the fact that $M\xi|_*$ is the sphere spectrum. This isomorphism is given by the Thom isomorphism (see [17], proof of Proposition 1). In this paper we will only use this edge homomorphism of the James spectral sequence for the stable homotopy, the identity map $id : X \rightarrow X$ and a given stable vector bundle $\xi : X \rightarrow BSO$. In this case $\epsilon\partial : \Omega_n(\xi) \rightarrow H_n(X, \mathbb{Z})$ is the map given by $[\nu : M \rightarrow X] \mapsto \nu_*[M]$ for every element $[\nu : M \rightarrow X] \in \Omega_n(\xi)$.

Notation 2.2. $E_{*,*}^*(\zeta)$ will denote the James spectral sequence for the stable homotopy as the generalized homology theory, identity fibration $id : X \rightarrow X$ and the stable bundle $\xi : X \rightarrow BSO$. The abbreviations JSS and AHSS will be used for the James and Atiyah-Hirzebruch Spectral sequences respectively. For any finite spectrum E , E_q^\wedge will denote the q -nilpotent completion of E at the prime q (sometimes called localization at \mathbb{Z}/q) (see [3]).

3. MAIN RESULTS

Let X be a simple Poincaré complex with dimension m . We impose the following condition on a stable vector bundle $\xi : X \rightarrow BSO$:

Condition 3.1. For each $r \leq m$ the differential in the James spectral sequence $d^r : E_{m,0}^r(\xi) \rightarrow E_{m-r,r-1}^r(\xi)$ is zero.

Observe that the image of the edge homomorphism in $H_n(X; \mathbb{Z})$ is the intersection of the kernels of all of the differentials that stem from $E_{m,0}^*$, i.e. $\text{im}(\mathfrak{e}\mathfrak{d}) = \bigcap_r \ker(d^r)$. Then Condition 3.1 implies that the group $E_{m,0}^2(\xi) = H_m(X; \mathbb{Z})$ is equal to $E_{m,0}^\infty(\xi)$, i.e. edge homomorphism is surjective. For a given class $[\nu : M \rightarrow X]$ in $\Omega_m(\xi)$ we have $\mathfrak{e}\mathfrak{d}[\nu : M \rightarrow X] = \nu_*[M]$. Thus, we can find a class $[\nu : M \rightarrow X]$ in $\Omega_m(\xi)$ such that $\mathfrak{e}\mathfrak{d}[\nu : M \rightarrow X] = \nu_*[M]$ is a generator of $H_m(X; \mathbb{Z})$ with the preferred orientation. As a result we get a degree one normal map $\nu : M \rightarrow X$, i.e. we have a surgery problem.

If Condition 3.1 does not hold for ξ , i.e. we have a non-trivial differential $d^r : E_{m,0}^r(\xi) \rightarrow E_{m-r,r-1}^r(\xi)$ for some r , then the edge homomorphism can not be surjective. This means $\nu_*[M]$ can not be a generator of $H_m(X; \mathbb{Z})$, i.e. ν can not be a degree one map. Hence, there is not a degree one normal map that represents a class in $\Omega_m(\xi)$. As a result there is not a manifold simple homotopy equivalent to X whose normal bundle lifts to X through ξ . Hence we have the following lemma:

Lemma 3.2. *A stable vector bundle ξ admits a degree one normal map if and only if Condition 3.1 holds for ξ .*

For the JSS for ξ , $E_{*,*}^*(\xi)$, there is a corresponding (isomorphic) AHSS for the Thom spectrum $M\xi$. For a given prime q , it is well known that the q -primary part of π_k^S is zero whenever $0 < k < 2q - 3$, see [19]. We use finiteness of π_k^S [14]. On each $(\text{mod } q)$ torsion part, the first non-trivial differentials of the AHSS are given by the duals of the stable primary cohomology operations [1]. Due to Wu formulas, when we pass to the JSS we need to know about the action of Steenrod algebra on the Thom class. For $p = 2$ the action of Steenrod squares on the Thom class $U \in H^*(M\xi; \mathbb{Z}/2)$ is determined by the Stiefel-Whitney classes. In fact the $(\text{mod } 2)$ Wu formula asserts that $Sq^i(U) = U \cup w_i$ (see [13], p.91).

Let S/q_* denote the homology represented by \mathbb{S}_q^\wedge , i.e. $S/q_*(E) = \pi_*(\mathbb{S}_q^\wedge \wedge E) = \pi_*(E_q^\wedge)$, where E is a finite spectrum. Consider the AHSS for the homology theory S/q , i.e. the coefficient groups will be $\pi_*(\mathbb{S}_q^\wedge)$. Due to naturality of the AHSS, the first non-zero differentials have to be stable primary cohomology operations independent of the generalized cohomology theory [1]. For each i with $0 < i < 2q - 3$ we have $\pi_i(\mathbb{S}_q^\wedge) = 0$ and $\pi_{2q-3}(\mathbb{S}_q^\wedge) = \mathbb{Z}/q$. Thus, the first non-trivial differential in this AHSS appears at the $(2q - 2)$ -page. This differential has to be a stable primary cohomology operation. The only $\text{mod } q$ operations in this range are 0 and dual of the $\text{mod } q$ Steenrod operation \mathcal{P}^1 . As in the proof of Lemma in [17, pp. 751], letting $E = \Sigma^{2q-2}H\mathbb{Z}/p$ as a test case one can see that d_{2q-2} is not always zero. The d^2 differential of $E_{*,*}^*(\xi)$ is given by the dual of the map $x \mapsto Sq^2(x) + w_2(\xi) \cup x$, see [17] Proposition 1. Let us write \mathbf{q}_1 for \mathbf{q}_1^q , where q is a fixed odd prime. We obtain a similar formula for the first non-zero differentials in $E_{*,*}^*(\xi)$ acting on $\text{mod } q$ torsion part.

Lemma 3.3. *For each $n \geq 2q - 2$ the differential on the $\text{mod } q$ part $d^{2q-2} : E_{n,0}^{2q-2}(\xi) \rightarrow E_{n-2q+2,2q-3}^{2q-2}(\xi)$ is equal to the dual of the map*

$$\delta : H^{n-2q+2}(X; \mathbb{Z}/q) \rightarrow H^n(X; \mathbb{Z}/q)$$

defined as $x \mapsto \mathcal{P}^1(x) + \mathbf{q}_1(\xi) \cup x$, composed with $\text{mod } q$ reduction.

Proof. Consider the AHSS for $M\xi$ and S/q . In this case the coefficient groups of the AHSS will be $\pi_*(\mathbb{S}_q^\wedge)$ and it will converge to $\pi_*(M\xi_q^\wedge)$. From above remarks, the

differential d_{2q-2} in the AHSS for $M\xi$ and S/q , is the dual of the mod q Steenrod operation

$$\mathcal{P}^1 : H^{n-2q+2}(M\xi, \mathbb{Z}/q) \rightarrow H^n(M\xi, \mathbb{Z}/q).$$

By the Thom isomorphism theorem an element of $H^*(M\xi, \mathbb{Z}/q)$ is of the form $U \cup \sigma$ where $\sigma \in H^*(X; \mathbb{Z}/q)$ and U is the Thom class. On the passage to the JSS the Cartan formula implies

$$\mathcal{P}^1(U \cup \sigma) = U \cup \mathcal{P}^1(\sigma) + \mathcal{P}^1(U) \cup \sigma = U \cup \mathcal{P}^1(\sigma) + U \cup \mathbf{q}_1(\xi) \cup \sigma$$

hence in the James spectral sequence these differentials become duals of the maps $\sigma \mapsto \mathcal{P}^1(\sigma) + \mathbf{q}_1(\xi) \cup \sigma$ composed with mod q reduction. \square

We have the following lemma for the differential d_m that stems from $E_{m,0}^m(\xi)$:

Lemma 3.4. *Let q be a prime and m be an odd number. Let X be a \mathbb{Z}/q -homology sphere with a given \mathbb{Z}/q -homology isomorphism $f : S^m \rightarrow X$. Then for any stable vector bundle $\xi : X \rightarrow BSO$ that is in the kernel of $f^* : K(X) \rightarrow K(S^m)$, the image of the differential d_m in $E_{0,m-1}^m(\xi)$ has trivial q -torsion.*

Proof. Let $\epsilon : S^m \rightarrow BSO$ be the stable vector bundle given by the composition $\xi \circ f$. The map $f : S^m \rightarrow X$ induces a map of spectra $Mf : M\epsilon \rightarrow M\xi$. Since f is a \mathbb{Z}/q -homology isomorphism, the induced map is also \mathbb{Z}/q -homology isomorphism, due to Thom isomorphism. Both $M\xi$ and $M\epsilon$ are connective spectrum and of finite type, and by assumption they are p -complete. Thus, the induced map on q -completions $Mf_q^\wedge : M\epsilon_q^\wedge \rightarrow M\xi_q^\wedge$ is a homotopy equivalence, see [3], Proposition 2.5 and Theorem 3.1. We have $f^*(\xi) = 0$ in $K(X)$. Hence ϵ is a trivial bundle. The Thom spectrum $M\epsilon$ is then homotopy equivalent to the wedge of spectra $S \vee \Sigma^m S$, as it is the suspension spectrum of $S^m \vee S^0$. Recall that $E_{*,*}^*(\epsilon)$ is the JSS for the identity fibration $S^m \rightarrow S^m$ and the trivial stable vector bundle. Hence $E_{*,*}^*(\epsilon)$ collapses on the second page, i.e. q -torsion in $E_{0,m-1}^m(\epsilon)$ survives to the $E_{0,m-1}^\infty(\epsilon)$. The result follows by comparing the q -torsion in $E_{0,m-1}^m(\epsilon)$, via Mf , with the q -torsion in $E_{0,m-1}^m(\xi)$. \square

Note that when X is a topological manifold and $m \neq 1 \pmod{8}$ we do not need to assume the existence of such an f . In this case there exists a degree one map $M \rightarrow S^m$ by Hopf's degree theorem and this map is a \mathbb{Z}/p -homology equivalence. We can compare spectral sequences by using this map as above.

Let $q = 2$ and $m = 1 \pmod{8}$ in Lemma 3.4. In this case f induces the identity on cohomology with coefficients $\mathbb{Z}/2$. Bott periodicity theorem asserts that $K(S^m) = K^{-m}(S^0) = \mathbb{Z}/2$. The map f induce a map on the Atiyah-Hirzebruch spectral sequences. At the second page we have $f^* : H^m(X; \mathbb{Z}/2) \rightarrow H^m(S^m; \mathbb{Z}/2)$, which is an isomorphism. The mod-2 class in $H^m(S^m; \mathbb{Z}/2)$ survives to the infinity page of the Atiyah-Hirzebruch spectral sequence for $K(S^m)$. Hence, f^* is a surjection on the infinity page by the naturality of the AHSS. Hence, we have the following remark:

Remark 3.5. In the case when $q = 2$ and $m = 1 \pmod{8}$ in Lemma 3.4, we have $[K(X) : \ker(f^*)] = 2$.

Suppose that $\zeta \notin \ker(f^*)$. Let $m = 1 \pmod{8}$ and $\mu : S^m \rightarrow BSO$ be the nontrivial element in $K(S^m)$. Suppose that Condition 3.1 holds for the JSS for μ and the identity on S^m . Then there is a degree one normal map $\nu : M \rightarrow S^m$

representing a class in $\Omega_m(\mu)$. Since every such map can be surgered to a simple homotopy equivalence [2], then we have $M \simeq S^m$. However, it is well known that every homotopy sphere is stably parallelizable, see [9]. Then we get a contradiction, as $\mu \circ \nu$ is nontrivial in $K(S^m)$. Now let $q = 2$ in the proof of Lemma 3.4. Then by comparing the 2-torsion in $E_{0,m-1}^m(\mu)$, via Mf , with the 2-torsion in $E_{0,m-1}^m(\xi)$, we get the following lemma:

Lemma 3.6. *In the case when $q = 2$ and $m = 1 \pmod{8}$ in Lemma 3.4, we have $\xi \in \ker(f^*)$ if and only if the image of the differential d_m in $E_{0,m-1}^m(\xi)$ has trivial q -torsion.*

3.1. Main theorem and its proof: Recall that $\mathfrak{R}(X)$ denotes the set of orbits of the natural action of $\text{Aut}_s(K(X))$ on $K(X)$, where $\text{Aut}_s(K(X))$ is the subgroup of automorphisms induced by the simple self equivalences of X (see Section 1). Let $\Phi : K(X) \rightarrow \mathfrak{R}(X)$ be the quotient map. We define a map Ψ from $\mathcal{M}(X)$ to $\mathfrak{R}(X)$ as follows: Let M be a smooth manifold equipped with a simple homotopy equivalence $\omega : M \rightarrow X$ and let η be the stable normal bundle of M . Let $g : X \rightarrow M$ be the homotopy inverse of ω . Then the pullback bundle $g^*(\eta)$ defines an element in $K(X)$. If $[M]$ is the diffeomorphism class of M in $\mathcal{M}(X)$, we define $\Psi([M]) := \Phi(g^*([\eta]))$.

Proposition 3.7. *Ψ is well defined.*

Proof. Let K be another manifold in the orbit $[M]$ with normal bundle κ , with a diffeomorphism $f : K \rightarrow M$ and with a simple homotopy equivalence $h : X \rightarrow K$. Since $f^*([\eta]) = [\kappa]$, we have $h^*f^*([\eta]) = h^*([\kappa])$. Since $\omega : M \rightarrow X$ is the homotopy inverse of g , we have $h^*([\kappa]) = h^*f^*([\eta]) = h^*f^*\omega^*g^*([\eta])$. Hence $h^*([\kappa])$ and $g^*([\eta])$ differ by an automorphism in $\text{Aut}_s(K(X))$ as the composition $\omega \circ f \circ h$ is homotopic to a simple self homotopy equivalence of X . By definition, in $\mathfrak{R}(X)$ they are the same. \square

Proof of Theorem 1.1. Let $[\xi]$ be an orbit in $\mathfrak{R}(X)_{(S, \mathbf{q}_1)}$ represented by $\xi : X \rightarrow BSO$. Consider the James spectral sequence, $E_{*,*}^*(\xi)$. Let q be a prime with $q \leq (m+1)/2$ and $q \notin S$. Then X is a \mathbb{Z}/q -homology sphere. By Lemma 3.4 the image of d_m has trivial q -torsion. Since $E_{m-r, r-1}^r(\xi) = H_{m-r}(X; \pi_{r-1}^S)$ does not contain q -torsion when $r < m$, image of the differentials $d_r : E_{m,0}^r(\xi) \rightarrow E_{m-r, r-1}^r(\xi)$ have trivial q -torsion as well. Hence, all of the differentials based at $E_{m,0}^r(\xi)$ have trivial q -torsion.

Now, let $p \in S$. Then $2p - 2 < m < 4p - 4$. It is well known that for $t < 4p - 5$ the p -torsion in π_t^S vanish except when $t = 2p - 3$, see for example [19], III Theorem 3.13 B. Since $2p - 2 < m < 4p - 4$, $E_{0,m-1}^m$ has trivial p -torsion, we have $d_m = 0$. Hence the only differential whose image can contain mod- p torsion appears at degree $2p - 2$. By Lemma 3.3, the differential d^{2p-2} on the p -torsion part is equal to the dual of the map

$$\delta : H^{m-2p+2}(X; \mathbb{Z}/p) \rightarrow H^m(X; \mathbb{Z}/p)$$

defined as $x \mapsto \mathcal{P}^1(x) + \mathbf{q}_1(\xi) \cup x$, composed with $(\text{mod } p)$ reduction. Let x be an element in $H^{m-2p+2}(X; \mathbb{Z}/p)$. By Poincaré duality, there exist an element s in $H^{2p-2}(X; \mathbb{Z}/p)$ such that $\mathcal{P}^1(x) = s \cup x$, see [7], Section 2. By definition $s = \mathbf{q}_1(X)$. Then d^{2p-2} is trivial on mod- p torsion as ξ is an element in $K(X)_{(S, \mathbf{q}_1)}$, i.e. as $\mathbf{q}_1^p(X) + \mathbf{q}_1^p(\xi) = 0$.

Now assume that $\mathbf{q}_1^p(X) + \mathbf{q}_1^p(\xi) \neq 0$. Then by Poincaré duality there exist an element $a \in H^{m-2p+2}(X; \mathbb{Z}/p)$ such that $a \cup (\mathbf{q}_1^p(X) + \mathbf{q}_1^p(\xi))$ is a generator of $H^m(X; \mathbb{Z}/p)$, i.e. $d^{2p-2}(a) \neq 0$.

As a result, Condition 3.1 holds for ξ if and only if $\xi \in K(X)_{(S, \mathbf{q}_1)}$. By Lemma 3.2 $\xi \in K(X)_{(S, \mathbf{q}_1)}$ if and only if ξ admits a degree one normal map. Since both $|\pi_1(X)|$ and m are odd, due to Theorem 1 in [2], the surgery obstruction groups vanish. Hence, we can do surgery and obtain a smooth manifold M with a simple homotopy equivalence $\omega : M \rightarrow X$ that represent a class in $\Omega_m(\xi)$ if and only if $\xi \in K(X)_{(S, \mathbf{q}_1)}$. By Lemma 3.4, the image of Ψ consists of orbits in $\mathfrak{R}(X)_{(S, \mathbf{q}_1)}$ represented by elements in the kernel of $f^* : K(X) \rightarrow K(S^m)$.

In the case when $m \not\equiv 1 \pmod{8}$ we have $K(S^m) = 0$ by Bott periodicity, i.e. image of Ψ is $\mathfrak{R}(X)_{(S, \mathbf{q}_1)}$. If $S = \emptyset$ then $\mathfrak{R}(X)_{(S, \mathbf{q}_1)} = \mathfrak{R}(X)$, i.e. Ψ is a surjection. This completes the proof. \square

From [2], it can be seen that the proof also implies that a version of this theorem is true if we omit the word “simple” at all. We have the following corollary:

Corollary 3.8. *Under the assumptions of Theorem 1.1 together with $m \not\equiv 1 \pmod{8}$ and $S = \emptyset$, the edge map $\bar{\epsilon}\bar{d} : \pi_*(\mathbb{S}) \rightarrow \pi_*(M\xi)$ is an inclusion for any stable vector bundle $\xi : X \rightarrow BSO$.*

Proof. As in the proof of Theorem 1.1 for a prime q the first differential in $E_{*,*}^*(\xi)$ that acts non-trivially on q -torsion appears in dimension $2q - 2$. Since $S = \emptyset$, X is a \mathbb{Z}/q -homology sphere for every prime q with $2q - 2 < m$. Hence, $d_r = 0$ for every $r < m$ and $d_m = 0$ by Lemma 3.4. Therefore, the first nontrivial differential in appear when $r \geq m + 1$, but then the target of d_r should be zero. Hence, $E_{*,*}^*(\xi)$ collapses at the second page, and we get that $\bar{\epsilon}\bar{d} : \pi_*(\mathbb{S}) \rightarrow \pi_*(M\xi)$ is inclusion. \square

The degree of the map f in the Theorem 1.1 plays an important role here, as it has to be co-prime to smaller primes. One can ask that what are the necessary and sufficient conditions on the pair (X, ξ) so that the natural map $\bar{\epsilon}\bar{d} : \pi_*^s \rightarrow \pi_*(M\xi)$ induced by the inclusion of the point is injective. It is well known that it is not the case for classical Thom spectra like MO or MSO (see [18]). When ξ is a trivial bundle then there are examples it is true. Another question, again regarding Corollary 3.8 may be for which spaces X , this natural map $\pi_*^s \rightarrow \pi_*(M\xi)$ is injective for every stable vector bundle $\xi : X \rightarrow BSO$. Corollary 3.8 provides just one such example.

3.2. Examples. Let $L^k(n)$ denote the quotient space $S^{2k+1}/\mathbb{Z}/n$ of the free linear action of \mathbb{Z}/n on S^{2k+1} . For a prime $p > 3$ the real topological K -theory of $L^k(p)$ is computed in [8]. Let $L^k(n, \mu)$ denote the orbit space of a free action μ of \mathbb{Z}/n on S^{2k+1} where μ acts by homeomorphisms. Such $L^k(n, \mu)$ are often called fake lens spaces, see [12] for more details. For a μ there is a lens space $L^k(n)$ homotopy equivalent to $L^k(n, \mu)$, see [5] P.456. For a prime p the group of homotopy classes of self homotopy equivalences of $L^k(p, \mu)$ is given as follows: $\text{Aut}(L^k(p, \mu))$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{Z}/p) = \mathbb{Z}/p^*$ which consists of elements t with $t^{k+1} = \pm 1 \pmod{p}$, see Theorem 2A in [15]. For $k = p - 3$ we have $t^{p-2} = \pm 1 \pmod{p}$, implying $t^{p-1} = \pm t \pmod{p}$. But since $t^{p-1} = 1 \pmod{p}$, we have that $t = \pm 1$. Same is also true for $k = 2p - 4$. Hence, in the case when $k = p - 3$ or $k = 2p - 4$, $\text{Aut}_s(K(L^k(p, \mu)))$ contains only 2-elements, namely the

identity and the automorphism mapping an element to its algebraic inverse. These are the easiest cases.

Assume first $k = p - 3$ and not divisible by 4. In this case, each orbit (except 0) in $\mathfrak{K}(L^k(p, \mu))$ contains two elements and hence we have $(|K(L^k(p, \mu))| + 1)/2 = (p^{(p-3)/2} + 1)/2$ elements in the set $\mathfrak{K}(L^k(p, \mu))$ (see [8], Lemma 3.3). By Theorem 1.1 the map Ψ is surjective, so that this number is a lower bound for the order of $\mathcal{M}(L^k(p, \mu))$. Suppose that $4|k$. We know the kernel of $f^* : K(L^k(p)) \rightarrow K(S^m)$ contains half of the elements in $K(L^k(p, \mu))$. Since $|K(L^k(p, \mu))| = 2 \cdot p^{(p-3)/2}$ due to Theorem 2 in [8], we have $(p^{(p-3)/2} + 1)/2$ elements in the image of Ψ . The same number is again a lower bound for the order of $\mathcal{M}(L^k(p, \mu))$.

Suppose now that $k = 2p - 4$ (so that k is not divisible by 4). We assume this time that $L^k(p)$ is the standard lens space, i.e. the lens space obtained by considering S^{2k+1} as sphere of \mathbb{C}^{k+1} with the action is given by multiplication of each coordinate by the p -th root of unity. Let $\bar{\sigma} = p^*(\gamma_k)$, where γ_k is the canonical complex line bundle over $\mathbb{C}P^k$ and $p : L^k(p) \rightarrow \mathbb{C}P^k$ is the natural projection. Due to Lemma 4.7 in [8], the tangent bundle of this manifold is represented by $\tau := (k + 1)\bar{\sigma}$. Then $\mathbf{q}_1(L^k(p)) = \mathbf{q}_1(\tau)$, by definition. For any bundle ζ with $\mathbf{q}_1(\zeta) = 0$ we have $\mathbf{q}_1(\zeta \oplus \tau) + \mathbf{q}_1(L^k(p)) = 0$. Hence, $\zeta \oplus \tau$ is an element in $K(L^k(p))_{\mathbf{q}_1}$. By using Theorem 2 in [8] we can see that $K(L^k(p))$ contains $(p - 3)/2$ -fold direct sum of \mathbb{Z}/p^2 . For each element ξ in these summands, we have $\mathbf{q}_1(p\xi) = 0$ where $p\xi$ denote the p -fold Whitney sum of ξ . Hence, there are at least $p(p - 3)/2$ elements in $K(L^k(p))$ with $\mathbf{q}_1 = 0$. Together with the fact that $k + 1$ is relatively prime to p . There are at least $p(p - 3)/2$ elements in the set $K(L^k(p))_{\mathbf{q}_1}$. Since $k = 2p - 4$, as before $\text{Aut}_s(K(L^k(p)))$ contains only two elements (by Theorem 2A in [15] combined with Fermat's little theorem). Hence, under the action of $\text{Aut}_s(K(L^k(p)))$ each orbit in $K(L^k(p))_{\mathbf{q}_1}$ consists of at most 2 elements. Hence $\mathfrak{K}(L^k(p))_{\mathbf{q}_1}$ contains at least $\lfloor p(p - 3)/4 \rfloor$ elements, where $\lfloor - \rfloor$ denote the floor function. By Theorem 1.1 this is a lower bound for the order of $\mathcal{M}(L^k(p))$. Note that same procedure works as well if $L^k(p)$ is a smooth fake lens space, i.e. a quotient of a smooth action of \mathbb{Z}/n on S^{2k+1} , provided that its tangent bundle has order p^2 . In this case the same number is also a lower bound for the number of elements in $\mathcal{M}(L^k(p))$.

More generally let π be a group with $p \geq 3$ being the smallest prime dividing $|\pi|$. Let Σ be a homotopy m -sphere with $m \geq 5$. Suppose that there is a continuous free action of π on Σ and let $X = \Sigma/\pi$ so that $f : \Sigma \rightarrow X$ is a principal π -bundle. Then there is a map $\varphi : X \rightarrow B\pi$ that classifies f . The group of self equivalences $\text{Aut}(X)$ of X contains a normal subgroup isomorphic to all inner automorphism $\text{Inn}(\pi)$ of π (see [15] Corollary 1 and Theorem 1.4). Note that an inner automorphism induces the identity on all cohomology groups of $B\pi$. Let $\alpha : X \rightarrow X$ in $\text{Aut}(X)$ be a self equivalence. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & B\pi \\ \alpha \downarrow & & \downarrow \alpha_* \\ X & \xrightarrow{\varphi} & B\pi \end{array}$$

so that α and α_* induce same map on $\pi_1(X) = \pi$. Hence, by an argument as in [6] Theorem 7.26, the diagram commutes up to homotopy. Hence by comparing

the Atiyah-Hirzebruch spectral sequences via φ we get that the inner automorphisms of π induce identity on $K(X)$ as well (in most cases $\text{Out}(\pi)$ is much smaller than $\text{Aut}(\pi)$). As a result we get $\text{Aut}_s(K(X))$ contains only the automorphisms induced by the outer automorphisms. Let us denote $\text{Out}(X) = \text{Aut}(X)/\text{Inn}(\pi)$ the group of outer self-equivalences of X . In this case $\mathfrak{K}(X)$ contains at least $\lfloor |K(X)|/|\text{Out}(X)| + 1 \rfloor$ element.

Assume first $m \not\equiv 1 \pmod{8}$ and $m < 2p - 2$, then by Theorem 1.1 we have $\Psi : \mathcal{M}(X) \rightarrow \mathfrak{K}(X)$ is surjective, i.e. $\lfloor |K(X)|/|\text{Out}(X)| + 1 \rfloor$ is a lower bound for the number of elements in $\mathcal{M}(X)$. Assume $m < 4p - 4$ and no other prime between p and $2p$ divides the order of π . Then the image of Ψ contains $\lfloor |pK(X)|/|\text{Out}(X)| \rfloor$ is a lower bound for the number of elements in $\mathcal{M}(X)$. If S is the set of primes between p and $2p$ which divide the order of π , then the image of Ψ contains $\mathfrak{K}(X)_{(S, q_1)}$, which has more than $\lfloor |CK(X)|/|\text{Out}(X)| \rfloor$ elements, where C is the product of primes in S .

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